

# Stable $\mathbb{A}^1$ -homotopy and $R$ -equivalence

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## Abstract

We prove that existence of a  $k$ -rational point can be detected by the stable  $\mathbb{A}^1$ -homotopy category of  $S^1$ -spectra, or even a “rationalized” variant of this category.

## 1 Introduction

Suppose  $k$  is a field and  $X$  a smooth proper  $k$ -variety. By the Lang-Nishimura lemma [Nis55], one knows that existence of a  $k$ -point is a  $k$ -birational invariant. By a remark of Morel and Voevodsky, one also knows that existence of a  $k$ -rational point is an unstable  $\mathbb{A}^1$ -homotopy invariant; see, e.g., [MV99, §3 Remark 2.5], where it is observed that this is a consequence of the fact that the Nisnevich topology is used in the construction of the unstable  $\mathbb{A}^1$ -homotopy category. The purpose of this note is to, in a sense, combine the two results above and to show that the ability to detect rational points persists in the Morel-Voevodsky stable  $\mathbb{A}^1$ -homotopy category of  $S^1$ -spectra as well as in Morel’s  $\mathbb{A}^1$ -derived category and even the  $\mathbb{A}^1$ -derived category with  $\mathbb{Q}$ -coefficients. Very loosely speaking, our results say that existence of rational points can be detected by purely cohomological means.

Write  $\mathbf{SH}_k^s$  for the Morel-Voevodsky stable  $\mathbb{A}^1$ -homotopy category of  $S^1$ -spectra (see [Mor05, Definition 4.1.1] for a precise definition). Let  $\Sigma_s^\infty X_+$  denote the  $\mathbb{A}^1$ -localization of the simplicial suspension spectrum of  $X$  with a disjoint basepoint attached. The 0-th  $S^1$ -stable  $\mathbb{A}^1$ -homotopy sheaf of  $X$ , denoted  $\pi_0^s(X_+)$ , is the Nisnevich sheaf associated with the presheaf on  $\mathcal{S}m_k$

$$U \longmapsto \mathrm{Hom}_{\mathbf{SH}_k^s}(\Sigma_s^\infty U_+, \Sigma_s^\infty X_+).$$

The structure morphism  $X \rightarrow \mathrm{Spec} k$  induces a morphism of sheaves  $\pi_0^s(X_+) \rightarrow \pi_0^s \mathrm{Spec} k_+$ . The sheaf  $\pi_0^s(X_+)$  is a birational invariant of smooth, proper  $k$ -varieties. If  $X$  has a  $k$ -rational point, the map  $\pi_0^s(X_+) \rightarrow \pi_0^s(\mathrm{Spec} k_+)$  is a split epimorphism. We prove a converse to this statement.

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**Theorem 1.** *If  $X$  is a smooth proper  $k$ -variety, then the following conditions are equivalent:*

- i)  $X$  has a  $k$ -rational point,*
- ii) there is a stable  $\mathbb{A}^1$ -homotopy class of maps  $\Sigma_s^\infty \operatorname{Spec} k_+ \rightarrow \Sigma_s^\infty X_+$  splitting the structure map  $\Sigma_s^\infty X_+ \rightarrow \Sigma_s^\infty \operatorname{Spec} k_+$ , and*
- iii) the morphism of sheaves  $\pi_0^s(X_+) \rightarrow \pi_0^s(\operatorname{Spec} k_+)$  is a split epimorphism.*

That (i)  $\implies$  (ii)  $\implies$  (iii) is clear, and the work goes into showing (iii)  $\implies$  (i). Proposition 2.1 allows us to show that  $\pi_0^s(\operatorname{Spec} k)_+ = \mathbb{Z}$ , so (iii) can be made more explicit. For a corresponding statement with  $\mathbb{Q}$ -coefficients see Remark 2.10. To put this result in context, we observe how these results combined with those of [AH10] give a framework for comparing rational points and 0-cycles of degree 1.

*Remark 2.* Let  $\Sigma_{\mathbb{P}^1}$  denote the operation of smashing with the simplicial suspension spectrum of  $(\mathbb{P}^1, \infty)$ , and let  $\Omega_{\mathbb{P}^1}$  be the adjoint looping functor. If  $E$  is any  $S^1$ -spectrum, there is a map  $E \rightarrow \Omega_{\mathbb{P}^1} \Sigma_{\mathbb{P}^1} E$ . We can iterate this functor to obtain a tower

$$E \longrightarrow \Omega_{\mathbb{P}^1} \Sigma_{\mathbb{P}^1} E \longrightarrow \Omega_{\mathbb{P}^1}^2 \Sigma_{\mathbb{P}^1}^2 E \cdots$$

The 0-th  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -homotopy sheaf of  $E$ , denoted  $\pi_0^{s\mathbb{A}^1}(E)$ , can be computed by means of the formula

$$\pi_0^{s\mathbb{A}^1}(E) = \operatorname{colim}_n \pi_0^s(\Omega_{\mathbb{P}^1}^n \Sigma_{\mathbb{P}^1}^n E).$$

The structure map  $X \rightarrow \operatorname{Spec} k$  induces a morphism  $\pi_0^{s\mathbb{A}^1}(X_+) \rightarrow \pi_0^{s\mathbb{A}^1}(\operatorname{Spec} k_+)$ . One says that  $X$  has a rational point up to stable  $\mathbb{A}^1$ -homotopy if the latter map is a split epimorphism. By [AH10, Theorem 1], if  $k$  is an infinite perfect field having characteristic unequal to 2, we know that a smooth proper  $k$ -scheme  $X$  has a 0-cycle of degree 1 if and only if it has a rational point up to stable  $\mathbb{A}^1$ -homotopy. Thus, under the stated hypotheses on  $k$ , the difference between a 0-cycle of degree 1 and  $k$ -rational point is measured by the difference between  $S^1$ -stable and  $\mathbb{P}^1$ -stable  $\mathbb{A}^1$ -homotopy theory. The existence of such a connection between rational points and 0-cycles of degree 1 was suggested in [Lev10, p. 395-6].

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## 2 Proof of Theorem 1

Let us introduce some notation for the rest of the note. Throughout, suppose  $k$  is a field. Let  $\mathcal{S}m_k$  denote the category of schemes that are separated, smooth, and have finite type over  $\operatorname{Spec} k$ . Write  $\mathcal{S}pc_k$  for the category of simplicial Nisnevich sheaves of sets on  $\mathcal{S}m_k$ ; objects

of this category will be called spaces. We identify Nisnevich sheaves with the corresponding simplicial objects.

Write  $\mathcal{H}(k)$  for the Morel-Voevodsky unstable  $\mathbb{A}^1$ -homotopy category. This category is constructed from the category  $\mathcal{Spc}_k$  by localizing at the class of  $\mathbb{A}^1$ -weak equivalences (see [MV99, §3.2]). We write  $\mathbf{SH}_k^s$  for the stable  $\mathbb{A}^1$ -homotopy category of  $S^1$ -spectra, e.g., as defined in [Mor05, §5]. Loosely speaking, this category is obtained from  $\mathcal{H}(k)$  by formally inverting the simplicial suspension operation. We write  $\mathbf{SH}_k$  for the stable  $\mathbb{A}^1$ -homotopy category of  $\mathbb{P}^1$ -spectra, e.g., as defined in [Jar00]; this category is obtained from  $\mathbf{SH}_k^s$  by further inverting the operation of smashing with the suspension spectrum of  $\mathbb{G}_m$ .

Recall that a presheaf of sets  $\mathcal{F}$  on  $\mathcal{S}m_k$  is called  $\mathbb{A}^1$ -invariant, if for any smooth scheme  $U$  the map  $\mathcal{F}(U) \rightarrow \mathcal{F}(U \times \mathbb{A}^1)$  induced by pullback along the projection  $U \times \mathbb{A}^1 \rightarrow U$  is a bijection. If  $\mathcal{X}$  is any space, we write  $\mathbb{Z}(\mathcal{X})$  for the simplicial sheaf of abelian groups freely generated by the simplices of  $\mathcal{X}$ . The normalized chain complex of  $\mathbb{Z}(\mathcal{X})$ , for which we will write  $C_*(\mathbb{Z}(\mathcal{X}))$ , is a chain complex of sheaves of abelian groups.

Write  $D(\mathcal{A}b_{Nis}(k))$  for the (unbounded) derived category of Nisnevich sheaves of abelian groups on  $\mathcal{S}m_k$ . A complex of sheaves of abelian groups  $A$  on  $\mathcal{S}m_k$  is called  $\mathbb{A}^1$ -local if for any complex  $B$  the map

$$\mathrm{Hom}_{D(\mathcal{A}b_{Nis}(k))}(B, A) \longrightarrow \mathrm{Hom}_{D(\mathcal{A}b_{Nis}(k))}(B \otimes \mathbb{Z}(\mathbb{A}^1), A)$$

is a bijection. A sheaf  $\mathcal{F}$  of abelian groups is said to be *strictly*  $\mathbb{A}^1$ -invariant if it is  $\mathbb{A}^1$ -invariant viewed as a complex of sheaves situated in degree 0. Consider the full subcategory of  $D(\mathcal{A}b_{Nis}(k))$  consisting of  $\mathbb{A}^1$ -local complexes; the inclusion of this subcategory into  $D(\mathcal{A}b_{Nis}(k))$  admits a left adjoint  $L_{\mathbb{A}^1}$  called the functor of  $\mathbb{A}^1$ -localization [CD09, Proposition 4.3]. Morel's  $\mathbb{A}^1$ -derived category  $D_{\mathbb{A}^1}(k)$  is (equivalent to) the full subcategory of the derived category of Nisnevich sheaves of abelian groups consisting of  $\mathbb{A}^1$ -local complexes.

Set  $C_*^{\mathbb{A}^1}(\mathcal{X}) := L_{\mathbb{A}^1} C_*(\mathbb{Z}(\mathcal{X}))$ ; this complex is called the  $\mathbb{A}^1$ -chain complex of  $\mathcal{X}$ . The 0-th  $\mathbb{A}^1$ -homology sheaf of  $\mathcal{X}$ , denoted  $\mathbf{H}_0^{\mathbb{A}^1}(\mathcal{X})$ , is just the 0-th homology sheaf of  $C_*^{\mathbb{A}^1}(\mathcal{X})$ . The functor  $\mathcal{X} \mapsto C_*^{\mathbb{A}^1}(\mathcal{X})$  induces a functor  $\mathcal{H}(k) \rightarrow D_{\mathbb{A}^1}(k)$ . The suspension isomorphism for homology shows that this functor factors through a functor  $\mathbf{SH}_k^s \rightarrow D_{\mathbb{A}^1}(k)$  that we will call abelianization. For recollections about the  $\mathbb{A}^1$ -derived category, see [Mor06, §3.2].

### The Hurewicz homomorphism

The abelianization functor induces a Hurewicz morphism  $\pi_0^s(\mathcal{X}_+) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(\mathcal{X})$  (note: the definition of  $\pi_0^s(-)$  given in the introduction makes sense for any  $\mathcal{X} \in \mathcal{Spc}_k$ ). The following result is a consequence of the stable  $\mathbb{A}^1$ -connectivity theorem [Mor05, Theorem 6.1.8], which states that  $(-1)$ -connected spectra or complexes are preserved by  $\mathbb{A}^1$ -localization.

**Proposition 2.1.** *If  $\mathcal{X}$  is a space, the canonical morphism  $\pi_0^s(\mathcal{X}_+) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(\mathcal{X})$  is an isomorphism of strictly  $\mathbb{A}^1$ -invariant sheaves.*

Because of this proposition, we can (and will) replace the 0-th stable  $\mathbb{A}^1$ -homotopy sheaf by the 0-th  $\mathbb{A}^1$ -homology sheaf of a space in the sequel. The next result follows immediately from Proposition 2.1 and, e.g., [Aso10, Theorem 2.2.9].

**Corollary 2.2.** *If  $k$  is an infinite field, the sheaf  $\pi_0^s(X_+)$  is a birational invariant of smooth and proper  $k$ -varieties.*

## 2.1 Strict $\mathbb{A}^1$ -invariance and birationality

**Definition 2.3.** Suppose  $\mathcal{F}$  is a presheaf of sets on  $\mathcal{S}m_k$ . We say  $\mathcal{F}$  is *birational* if for any open dense immersion  $U \rightarrow U'$  in  $\mathcal{S}m_k$ , the map  $\mathcal{F}(U') \rightarrow \mathcal{F}(U)$  is an isomorphism.

In the following lemma, we summarize some technical properties of birational presheaves. This result is “well known to the experts” and we include it for the convenience of the reader; results along these lines can also be found in, e.g., [Lev10, §2].

**Lemma 2.4.** *If  $\mathcal{F}$  is a birational presheaf, the free presheaf of abelian groups  $\mathbb{Z}(\mathcal{F})$  is also birational, and both  $\mathcal{F}$  and  $\mathbb{Z}(\mathcal{F})$  are Nisnevich sheaves. If  $\mathcal{F}$  is furthermore  $\mathbb{A}^1$ -invariant, then  $\mathbb{Z}(\mathcal{F})$  is Nisnevich flasque, and  $\mathbb{Z}(\mathcal{F})$  is strictly  $\mathbb{A}^1$ -invariant.*

*Proof.* To show that  $\mathcal{F}$  is a Nisnevich sheaf is, we just have to check that  $\mathcal{F}$  takes an elementary distinguished square

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \psi \\ U & \longrightarrow & X, \end{array}$$

(where  $\psi$  is étale,  $U \rightarrow X$  is an open immersion,  $X \setminus U$  is given the usual reduced scheme structure, and the map  $\psi^{-1}(X \setminus U) \rightarrow X \setminus U$  is an isomorphism) to a cartesian square. Since  $\mathcal{F}$  is birational, both the bottom and top maps are isomorphisms and so the diagram is cartesian. Now, if  $\mathcal{F}$  is birational, then by definition  $\mathbb{Z}(\mathcal{F})$  is also birational, and by what we just showed  $\mathbb{Z}(\mathcal{F})$  is also a Nisnevich sheaf.

If  $\mathcal{F}$  is also  $\mathbb{A}^1$ -invariant, it follows immediately that  $\mathbb{Z}(\mathcal{F})$  is also  $\mathbb{A}^1$ -invariant. We will now show that  $\mathbb{Z}(\mathcal{F})$  is Nisnevich flasque. To see this, recall that the Nisnevich cohomology can be computed by means of Čech cochains: [MV99, p. 95] mentions this without proof, but the proof is essentially identical to the corresponding statement in the étale topology; one uses Čech-derived functor spectral sequence and the fact [Nis89, Lemma 1.18.1] that the higher cohomology sheaves of a Nisnevich sheaf of abelian groups vanish. Therefore, suppose  $X$  is an irreducible smooth scheme,  $u : U \rightarrow X$  is a Nisnevich cover of  $X$ . By lifting the generic point  $\eta$  of  $X$ , we can find a component of  $U$  that is birational to  $X$ . Since each map  $U^{\times n+1} \rightarrow U^{\times n}$  is also a Nisnevich cover, it follows that  $\mathbb{Z}(\mathcal{F})(U^{\times n}) \rightarrow \mathbb{Z}(\mathcal{F})(U^{\times n+1})$  is injective and thus all higher Nisnevich cohomology of  $\mathbb{Z}(\mathcal{F})$  vanishes.  $\square$

**Corollary 2.5.** *If  $\mathcal{F}$  is a birational and  $\mathbb{A}^1$ -invariant sheaf of sets, the canonical map  $\mathcal{F} \rightarrow \mathbb{Z}(\mathcal{F})$  induces an isomorphism  $\mathbf{H}_0^{\mathbb{A}^1}(\mathcal{F}) \rightarrow \mathbb{Z}(\mathcal{F})$ .*

*Proof.* By definition  $\mathbf{H}_0^{\mathbb{A}^1}(\mathcal{F}) = H_0(L_{\mathbb{A}^1}\mathbb{Z}(\mathcal{F}))$ . However, since  $\mathbb{Z}(\mathcal{F})$  is Nisnevich flasque, it follows that  $\mathbb{Z}(\mathcal{F})$  is  $\mathbb{A}^1$ -local, i.e., the canonical map  $L_{\mathbb{A}^1}(\mathbb{Z}(\mathcal{F})) \rightarrow \mathbb{Z}(\mathcal{F})$  is an isomorphism.  $\square$

*Example 2.6.* Suppose  $X$  is an  $\mathbb{A}^1$ -rigid smooth proper  $k$ -scheme (see [MV99, §3 Example 2.4]). Given an open dense immersion  $U \rightarrow U'$ , the map  $X(U') \rightarrow X(U)$  is an isomorphism; indeed any such map is uniquely determined by where it sends the generic point of each component. As a consequence  $\mathbb{Z}(X)$  is a strictly  $\mathbb{A}^1$ -invariant sheaf. Because  $\mathbb{Z}(X)$  is  $\mathbb{A}^1$ -local, we see that the canonical map  $C_*^{\mathbb{A}^1}(X) = L_{\mathbb{A}^1}\mathbb{Z}(X) \rightarrow \mathbb{Z}(X)$  is an isomorphism, and thus that  $\mathbf{H}_0^{\mathbb{A}^1}(X) = \mathbb{Z}(X)$ . Thus,  $X$ ,  $\mathbb{Z}(X)$ , and  $\mathbf{H}_0^{\mathbb{A}^1}(X)$  are all birational sheaves and by Lemma 2.4 all these sheaves are all strictly  $\mathbb{A}^1$ -invariant. As a consequence of Corollary 2.5, we deduce that if  $k$  is infinite and  $X'$  is any smooth proper variety that is stably  $k$ -birationaly equivalent to a smooth proper  $\mathbb{A}^1$ -rigid variety  $X$ , then  $\mathbf{H}_0^{\mathbb{A}^1}(X') = \mathbf{H}_0^{\mathbb{A}^1}(X)$ .

## 2.2 Birational connected components and the main result

Suppose  $X$  is a smooth proper variety over a field  $k$ . If  $L/k$  is a separable, finitely generated extension, recall that two  $L$ -points in  $X$  are  $R$ -equivalent if they can be connected by the images of a chain of morphisms from  $\mathbb{P}_L^1$  to  $X$  (over  $k$ ) [Man86]. There is a birational sheaf related to  $R$ -equivalence classes of points in  $X$ .

**Theorem 2.7.** *If  $X$  is a smooth proper  $k$ -variety, there is a birational and  $\mathbb{A}^1$ -invariant sheaf  $\pi_0^{b\mathbb{A}^1}(X)$  together with a canonical map  $X \rightarrow \pi_0^{b\mathbb{A}^1}(X)$  functorial for morphisms of proper varieties such that for any separable finitely generated extension  $L/k$  the induced map  $X(L) \rightarrow \pi_0^{b\mathbb{A}^1}(X)(L)$  factors through a bijection  $X(L)/R \rightarrow \pi_0^{b\mathbb{A}^1}(X)(L)$ .*

*Proof.* Everything except the statement of functoriality is included in [AM09, Theorem 6.2.1]. Since  $\pi_0^{b\mathbb{A}^1}(X)$  is a birational and  $\mathbb{A}^1$ -invariant sheaf, to construct a morphism  $\pi_0^{b\mathbb{A}^1}(Y) \rightarrow \pi_0^{b\mathbb{A}^1}(X)$ , it suffices to observe that by the definition of  $R$ -equivalence a morphism  $f : X \rightarrow Y$  induces morphisms  $X(L)/R \rightarrow Y(L)/R$  for every finitely generated separable extension  $L/k$ .  $\square$

If  $X$  is a smooth proper variety, we can consider the sheaf  $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$ . By Lemma 2.4, it follows that  $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$  is a strictly  $\mathbb{A}^1$ -invariant sheaf, and Corollary 2.5 gives rise to a canonical identification  $\mathbf{H}_0^{\mathbb{A}^1}(\pi_0^{b\mathbb{A}^1}(X)) \xrightarrow{\sim} \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$ . As a consequence of Theorem 2.7 we deduce the existence of a canonical morphism

$$\phi_X : \mathbf{H}_0^{\mathbb{A}^1}(X) \longrightarrow \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X)).$$

Because  $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$  is a strictly  $\mathbb{A}^1$ -invariant sheaf, existence of this morphism also follows immediately from [Aso10, Lemma 2.2.3], which states that  $\mathbf{H}_0^{\mathbb{A}^1}(X)$  is initial among strictly  $\mathbb{A}^1$ -invariant sheaves  $M$  admitting a morphism of sheaves  $X \rightarrow M$ .

*Remark 2.8.* It seems reasonable to expect that the morphism  $\phi_X : \mathbf{H}_0^{\mathbb{A}^1}(X) \rightarrow \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$  is an isomorphism. Since our goal is to get as quickly as possible to the connection with rational points we did not pursue this further.

**Corollary 2.9.** *If  $X$  is a smooth proper  $k$ -variety, then the set  $X(k)$  is non-empty if and only if the map  $\mathbf{H}_0^{\mathbb{A}^1}(X) \rightarrow \mathbb{Z}$  induced by the structure map is a split surjection.*

*Proof.* If  $X(k)$  is non-empty, then we get a morphism  $\mathbb{Z} = \mathbf{H}_0^{\mathbb{A}^1}(\mathrm{Spec} k) \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(X)$  that splits the map induced by the structure morphism. Conversely, note that the map  $\mathbf{H}_0^{\mathbb{A}^1}(X) \rightarrow \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$  is functorial in  $X$ , and thus the morphism  $\mathbf{H}_0^{\mathbb{A}^1}(X) \rightarrow \mathbb{Z}$  factors through the morphism  $\phi_X$ . A splitting  $\mathbb{Z} \rightarrow \mathbf{H}_0^{\mathbb{A}^1}(X)$  therefore gives rise to a non-trivial morphism  $\mathbb{Z} \rightarrow \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$ , i.e., an element of  $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))(k)$ . The group  $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))(k)$  is by Theorem 2.7 the free abelian group on the set  $X(k)/R$ . Since the group  $\mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))(k)$  is a non-trivial free abelian group, we deduce that  $X(k)/R$  has at least 1 element, and therefore  $X(k)$  is non-empty.  $\square$

*Proof of Theorem 1.* Combine Corollary 2.9 and Proposition 2.1.  $\square$

*Remark 2.10.* The rationalized  $\mathbb{A}^1$ -derived category is obtained by following the construction of the  $\mathbb{A}^1$ -derived category sketched above and replacing abelian groups by  $\mathbb{Q}$ -vector spaces throughout. Replacing  $\mathbb{Z}$  by  $\mathbb{Q}$  in all of the above allows one to deduce that existence of  $k$ -rational point is detected by the *rationalized*  $\mathbb{A}^1$ -derived category. To be precise, if  $X$  is a smooth proper  $k$ -scheme, then  $X$  has a  $k$ -rational point if and only if the canonical map  $\mathbf{H}_0^{\mathbb{A}^1}(X, \mathbb{Q}) \rightarrow \mathbb{Q}$  induced by the structure morphism  $X \rightarrow \mathrm{Spec} k$  is a split epimorphism. This statement also implies a statement about an appropriate “rational” version of the stable  $\mathbb{A}^1$ -homotopy category of  $S^1$ -spectra, but we leave this to the reader.

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